Signals with fractal characteristics and the Shannon-Whittaker theorem

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Abstract

We introduce two theorems which permit to obtain a fractal signal from the product superposition of periodic signals. These periodic components are band-limited functions and therefore, we can consider the conditions for the application of the Shannon-Whittaker theorem for the reconstruction of the fractal signal. Also, we relate such theorem with the order of the signal that is under study. We use an expression of the sampling theorem for periodic (band-limited) functions.

1. Introduction

In the last decade diverse works related with the processing of complex signals, and particularly with fractal structure [1-3] were developed, due to the variety of applications, for example in audio [4], communications [5, 6] and biomedicine [7, 8]. In this context the multiplication of basic signals and their obtained complexity can be important for their transmission, study or parameters determination. In such applications the word "prefractal" (as defined by Mandelbrot [9]) should be used strictly, when referring to a self-similar object with certain limitations. However, it is clear that the difference between "fractal" and "prefractal" exists only from the mathematical point of view.

Any signal, before being processed, should be measured or sampled. This means that a complex or fractal signal should have certain conditions on the sampling interval, different to the ones found in the periodic signal (for example). Just recently this aspect has become of interest for the reconstruction of certain signals or functions. The Shannon-Whittaker theorem (or sampling theorem) [10-13] relates the measured points of a certain signal and the possibility of its complete reconstruction, based on such measurement.

In previous works [14-16], it has been demonstrated that some fractal structures can be obtained starting from periodic distributions (with a scaling factor between them). This fact is important for applications in the processing of different types of signals, where a particular geometry can be required in the final signal. As an extension of these results, in this work we first include a development of a feasible and simple method for constructing complex structures with the superposition of domains distributed in a periodic way. Then, with two theorems we can observe that the results obtained for the cases of binary structures, for which digital signals are an example, can be extended for the case of structures with continuous variation, such as analog signals for example. The demonstrations for these theorems are related with the theory of IFS (Iterated Function System) [17-19]

Here, we deal mainly with a direct problem; that is to say, the characteristics of the original signal are known and we want to establish a way to reconstruct it. First, we demonstrate that the product superposition of functions or signals can give as a result a signal with fractal characteristics. Furthermore, we use a consequence, expressed for the case of periodic band-limited functions [20]. Then, our interest is the inclusion of this formulation for the case of fractal signals, obtained through a product superposition of periodic functions and consider if the Shannon-Whittaker theorem must be modified or adapted for such signals. This way, we want to show a consequence of the sampling theorem for the reconstruction of signals with complex geometry.

2. Mathematical foundations

There are three basic transformations for building fractal objects: change of scale, translation and rotation. In several works these transformations were used for the construction of fractal structures [14, 21, 22]. For these cases, we used periodic domains which are defined through the distribution of disjoined sets included in a 1D or 2D Euclidean space. The mathematical expression to obtain such fractal structures is:

$$C(x,y)=\prod_{k=1}^{N} P\left[s^{k};x,y\right] \qquad (1)$$

where P is a periodic function and s is the scaling function. Fig. 1 shows graphically the product of Eq. (1), and Fig. 2 shows an example for the construction of a triadic Cantor function, with periodic distribution, using Eq. (1).

2.1. Simple functions

A function $\boldsymbol{f}: \boldsymbol{X} \rightarrow \boldsymbol{R}$ is called simple if [22]:

$$f(x)=\sum_{i=1}^{K} R_i\, \chi\left[I_i;x\right], \text{ with } \chi\left[I_i;x\right]=\begin{cases}1 \text{ if } x\in I_i\\ 0 \text{ if } x\notin I_i\end{cases} \qquad (2)$$

being $\boldsymbol{I_i} \in \boldsymbol{B}$ (a Borel set), $\boldsymbol{R_i} \in \Re$ for $\boldsymbol{i}=1,2,\ldots,\boldsymbol{K}$ and $\bigcup_{\boldsymbol{i}=1}^{\boldsymbol{K}} \boldsymbol{I_i}=\boldsymbol{X}$ with $\boldsymbol{I_i}\cap \boldsymbol{I_j}=\Phi$ for $\boldsymbol{i}\neq \boldsymbol{j}$. Where χ is the characteristic function [23] of each set I_i. In this way, functions can be approximated from the obtained values in certain points. Clearly, R_i from Eq. (2) may be referred to the measured points for the function *f(x)*, which are taken into account according with the value of the characteristic function.

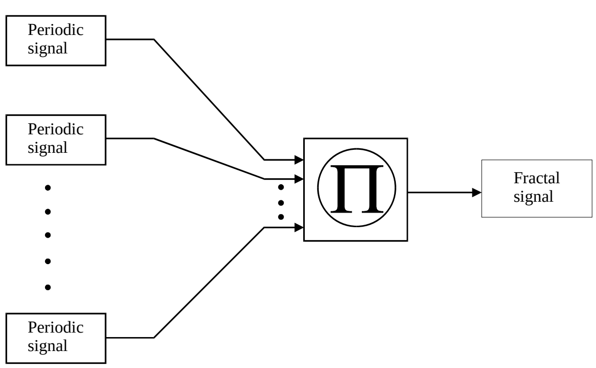

Figure 1. Graphic method to obtain a fractal signal using periodic components.

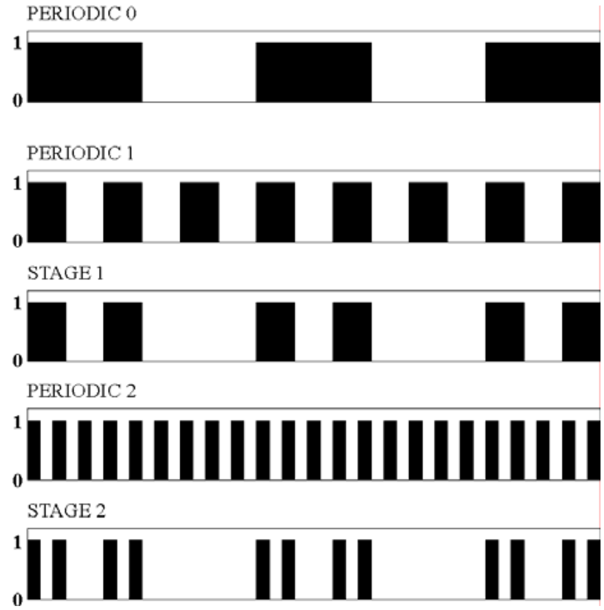

Figure 2. An example for the construction of periodic Cantor functions with fractal dimension *D≈0.6309*.

2.2. Fractal signals with periodic components

Now, we show a general case, and exemplify it graphically with cos^2 functions. For this, it is important to keep in mind the following theorems.

Theorem 1. If we have an iterative function, defined through:

$$\begin{aligned}&g^{k}(x,y)=T\left[s^{k};x,y\right]g^{k-1}(x,y)\\ &k=1,\ldots,K \text{ and } g^{0}(x,y)=T\left[s^{0};x,y\right]\end{aligned} \qquad (3)$$

where s is an integer value, T is a periodic function and k refers to each iteration. Then, $g^k(x)$ defines contractions in the Haussdorf space *H(X)* and each $g^k(x)$ allows to define a non-linear IFS given by $\{f^1,\ldots,f^P;P=P(s)\}$.

Proof. This theorem can be demonstrated when observing Fig. 3, and considering a sequence of sets, defined in the metric space *(X,d)*, whose boundary is the function $g^k(x)$ and the line for $y=0$. From Fig. 3 it is clear that for each k-component we have s periods in the region $[-L,L]$. Also, the total set for each iteration F^k (denoted with different grey levels, for F^0 F^1 F^2…) is defined as:

$$F^{k}=\left\{(x,y):\left|x\right|\leq L,\, y\leq g^{k}(x,y)\right\} \qquad (4)$$

which can be divided in subsets defined, as shown in Fig.3, with the following expression:

$$F^k_{[p_1\dots p_k]} = \{(x,y,z) : x \le |x_i(k) - x_{i+1}(k)|,\ y \le g^k(x,y) \qquad (5)$$
$$g^k(x_i(k)) = g^k(x_{i+1}(k)), \text{ with } i = I(p_k \dots p_1)$$

this is, $x_i(k), x_{i+1}(k)$ are two successive zeros of $g^k(x)$, where i is a function of the sequence $[p_k \dots p_1]$ which are related with the successive iterations and with the scaling factor. For each k-iteration, this means that a new set is obtained, which is included (but not equal) in the previous one. Then, given two points x_1, x_2 we have:

$$d\left(f^{p_k}(x_1), f^{p_k}(x_2)\right) \le \frac{1}{s^k} d(x_1, x_2) \qquad (6)$$
$$\forall\ x_1, x_2 \in F^{k-1}_{[p_{k-1}\dots p_1]}$$

and so, each component f^p_k defines contractions on the metric space (X,d) and $\{f^1, \dots, f^P; P=P(s)\}$ is an IFS on $(H(X),h(d))$, originated from the boundary defined by Eq. (2). Finally, it can be seen that F is an attractor of the sequence F^k, that is the result shown in Eq. (4): $F = \lim_{k \to \infty} F^k$.

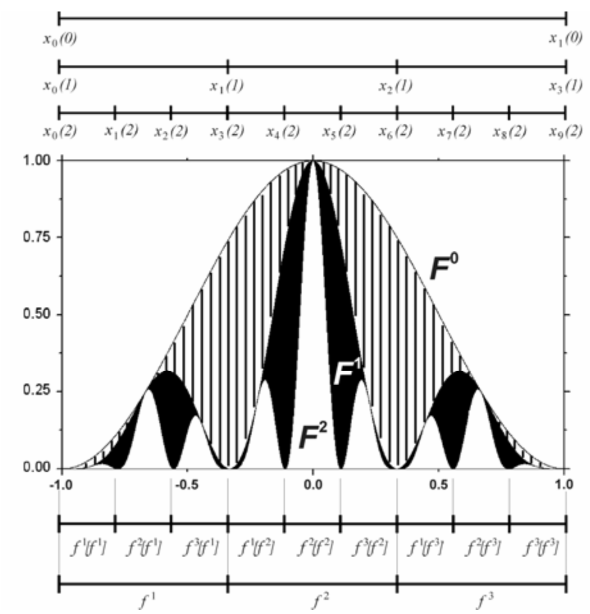

Figure 3. The product of periodic functions $C=cos^2$, where $F^0F^1F^2\dots$ are the sets obtained in each iteration and $x_i\ (k)$ are the points for $g^k(x)=0$ ($L=1$).

Theorem 2. In each iteration of Eq. (3) fixed points on the boundary and on the x-axis are obtained, which will tend to complete the total boundary of the set F^k as shown in Fig. 4, because for $k \to \infty$ we will have infinite fixed points. Then, these points belong to the final set F, which is a fixed point of $H(X)$ and the attractor of the IFS defined in Theorem 1.

Proof. According to Eq. (3) we have:

$$g^N(x,y) = \prod_{k=1}^{N} T\left[s^k; x, y\right] \qquad (7)$$

Given $k=M \le N$, every new fixed point x_F, for the M-iteration ($M \ge 1$), implies:

$$T\left[s^k; x_F, y_F\right] = 1 \Rightarrow (x_F, y_F) \text{ fixed point} \qquad (8)$$

Then, from Eq. (3) and using the result of Eq. (5) we have that, for the complete fractal (with $N>M$):

$$g^N(x,y) = g\{\dots\{g^{M-1}(x,y)\,T\left[s^k; x, y\right]\}\} = \prod_{k=1}^{M-1} T\left[s^k; x, y\right] \qquad (9)$$

and we finally obtain that $g^{M-1}(x)$ is a fixed point of the transformation defined by Eq. (7). In the case $s=3$, for $k=M$ the corresponding periodic component has 3^M fixed points on the boundary.

The fixed points on the boundary and how they are obtained for each iteration, from the product with periodic components, is also shown in Fig. 4. For this case, in the figure on the top we can see the three first periodic components ($s=3$) and the points where the crests of the function are equal to one.

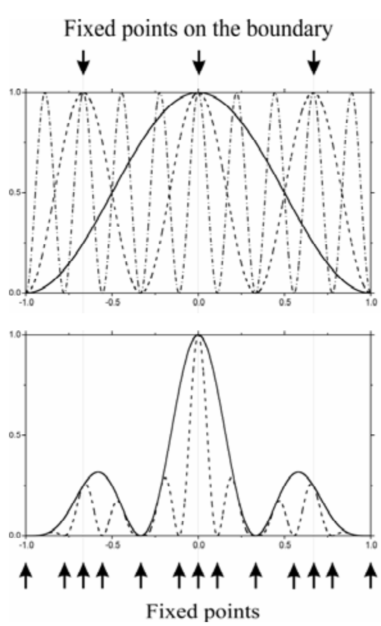

Figure 4. Fixed points for each k-iteration, introduced from the periodic components.

For the two first components only the central point is fixed (see Fig. 4) and, when the third component is included, we obtain other two fixed points. Furthermore, we must consider that there are fixed points on the axis x, for:

$$T\left\lfloor s^k ; x_F , y_F \right\rfloor = 0 \qquad (10)$$

which are important in order to assure the existence of mappings f^p $(p=1,\ldots,P(s))$, as was demonstrated in Theorem 1.

With the results obtained in the previous theorems, it is demonstrated that there is an IFS which permits the generation of a sequence F^k. Also, this sequence of sets has an attractor and the way in which the points of this attractor are obtained are shown in the second theorem. So, we have presented the superposition of periodic signals to obtain a fractal signal.

2.2. Measurement of complex signals

When the previous results are implemented, for the recording of complex signals, infinite number of points of the fractal objects are never obtained. In the measurement of a certain signal only discrete points are obtained. Then, we can relate these points with the representation as a simple function. For example, the functions in Fig. 5 are built with finite number of points, with a scaling factor between each periodic component. So, we want to obtain, from this finite number of points of the signal, the corresponding expression for the sampling theorem.

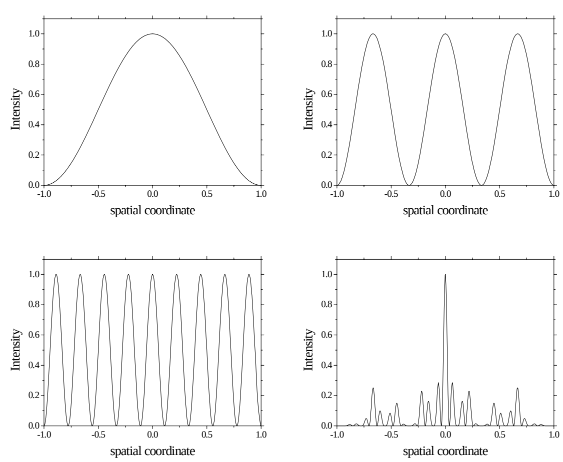

Figure 5. Complex signal obtained the product superposition of cos^2 components.

3. Simple functions and Shannon-Whittaker sampling theorem

Until now we have shown how to obtain structures from fractals by using periodic band-limited functions. The Shannon-Whittaker sampling theorem assures us that we have a good representation of a function which has an experimental base, since the function is represented by discrete points. We will use another version of the sampling theorem for the case of periodic functions. In [20] the expression for the sampling theorem was obtained, which is given by:

$$g(x) = \frac{\sin(2\pi\Lambda_x x)}{2K} \sum_{n=-L}^{M-1} (-1)^n F\left[\frac{n}{2\Lambda_x}\right] \times \qquad (11)$$

$$\times \left[(-1)^{K+1} \tan\left(2\pi\Lambda_x \frac{x - \frac{n}{2\Lambda_x}}{2K} \right) + \cot\left(2\pi\Lambda_x \frac{x - \frac{n}{2\Lambda_x}}{2K} \right) \right]$$

being Λ_x the sampling frequency, and $L+M=K$ are arbitrary integers (see Ref. [20]).

So, the sampling theorem is applied for the product superposition of periodic functions, which initially can be independently obtained. The set of points g_n can be represented through a simple function, as defined by Eq. (2). Then, using the product of functions (see Eq. (7)), the Shannon-Whittaker theorem can be expressed as:

$$g(x) = \frac{\sin(2\pi\Lambda_x x)}{2K} \prod_{k=1}^{N} \left\{ \sum_{n=-L}^{M-1} (-1)^n \left[R_n \chi[I_n ; x] \right]^{(k)} \times \right. \qquad (12)$$

$$\left. \times \left[(-1)^{K+1} \tan\left(2\pi\Lambda_x \frac{x - \frac{n}{2\Lambda_x}}{2K} \right) + \cot\left(2\pi\Lambda_x \frac{x - \frac{n}{2\Lambda_x}}{2K} \right) \right] \right\}^{h_k}$$

where $\chi[I_n;x]$ is the characteristic function for each periodic component, the interval I_n is related with the width of the corresponding sampling interval and the supra-index k indicates each periodic component.

This means (from the linear systems theory) that if there is a system with several inputs (one for each periodic component), a signal described by Eq. (7) at the output then, the sampling for each component and for the output signal is related through Eq. (12), as shown Fig. 6. The sampling interval will be given by the corresponding interval of the component with the smallest period.

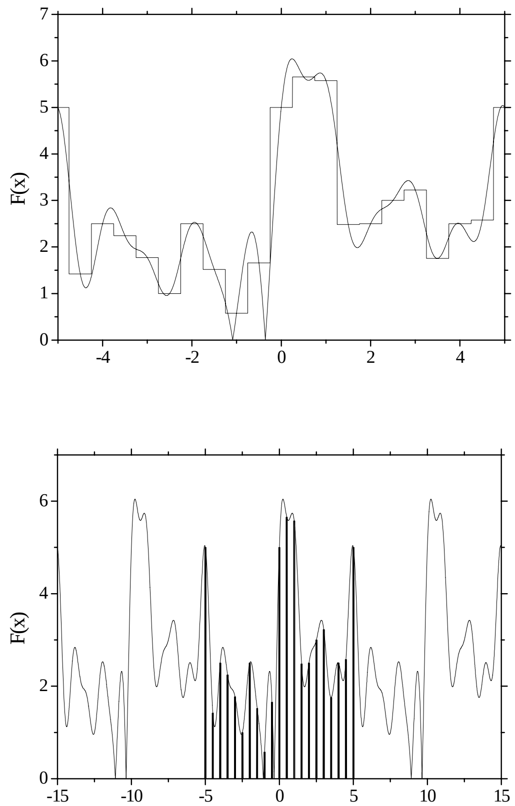

Figure 6. Simple function and the sampling theorem applied for an arbitrary signal.

4. Conclusions

We extend the method which uses periodic components to build digital fractal functions, for the case of analog fractal functions. We used the well-known results for fractal binary functions, for which the construction is made through a product of periodic functions (with values 0 and 1). As an example, cosine functions (cos^2) are used and the analog fractal signals are obtained. Then, we conclude that the results for discrete signals can be extrapolated for continuous functions. The fractal characteristics of such signals are based in two theorems that we included in this paper, which assure that there are attractors and fixed points for our method, in a similar way to the theory of dynamical systems.

Since the function has a periodic envelope, we use a version of the sampling theorem which permits us to represent it (and their scaled periodic components) from finite number of points. The sampling interval that must be used is the one corresponding to the component with the smallest period.

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