Controlled Lagrangian for the stabilization of an inverted pendulum.

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Abstract

A Lagrangian Controller for the stabilization of the inverted pendulum cart system is presented in this paper. The control strategy consists of finding a controller that transforms the closed-loop system into another Euler-Lagrange system with a fixed inertia matrix. This was done by solving two matching conditions related with the total energy of the closed-loop system. The resulting control strategy turns out to be locally asymptotically exponentially stable with a very large domain of attraction.

1. Introduction

A classic and challenging problem in control theory is the stabilization of the inverted pendulum cart system (**IPC**). This device consists of a free vertical rotating pendulum with a pivot point mounted on a cart. The cart can be moved horizontally by means of a horizontal force, which is the control of the system. Since the angular acceleration of the vertical pendulum cannot be directly controlled. The **IPC** has attracted the attention of several researchers as a benchmark for nonlinear control design (see [1] and references therein) and it is an interesting example of an under-actuated mechanical system. As consequence, many standard control strategies not are unable to control it. For example, the **IPC** is not input-output linearized by means of static feedback [2]. Besides, the system loses controllability and other geometric properties while the pendulum moves through the horizontal plane [3]. Fortunately, this system is locally controllable around the unstable equilibrium point, therefore the stabilization problem can be solved locally by a direct pole placement procedure [4]. In general, the stabilization problem of the **IPC** consists in bringing up the pendulum to the upright vertical position, with the cart resting at the origin. A detailed review of the state of the art of the problem here treated is beyond the scope of this work. However, we refer the interested reader to the following references: [5], [6], [7], [8], [9] and [10].

In this paper we develop a Controlled lagrangian for bringing the pendulum to the top position and the cart to the zero position, simultaneously. The proposed controller has two main advantages: the domain of stability can be as large as desired. As a matter of fact, this set is constituted for almost states for which the pendulum is initialized over the horizontal plane. The closed-loop system is robust with respect to small damping forces since the obtained closed-loop system is locally exponentially stable. Our main contribution is to solve, in an easy way, the two needful matching conditions, necessary to find one stabilizing controller for the **IPC**, without the necessity of solving a set of more complex **PDE**, as would be necessary if we used another kind of model matching [11] and [12]. The main difference between the presented model matching and the previous work presented by [8], is that we force the closed-loop system to behave like a stable Euler-Lagrange system with fixed inertia matrix, eliminating the need to solve a set of three nonlinear differential equations to get the variable inertia matrix, as was done in [8]. Furthermore, in that work the inertia matrix cannot be obtained explicitly.

The organization of this paper is as follows. Section 2 presents the dynamic model of the **IPC**. In section 3 a suitable model matching and the solution of the two matching conditions are presented. Section 4 depicts the stability analysis of the closed-loop system and Section 5 presents some computer simulations. Finally the conclusions are given in Section 6.

2. The inverted pendulum cart system

Consider a traditional **IPC** (see Figure 1), which is described by the following normalized set of differential equations

$$\begin{aligned} &\cos\theta\ddot{q} + \ddot{\theta} - \sin\theta + \beta\dot{\theta} = 0, \\ &(1+\delta)\ddot{q} + \cos\theta\ddot{\theta} - \dot{\theta}^2\sin\theta = f, \end{aligned} \tag{1}$$

where q is the cart normalized displacement, θ is the angle that the pendulum forms with the vertical, f is the force

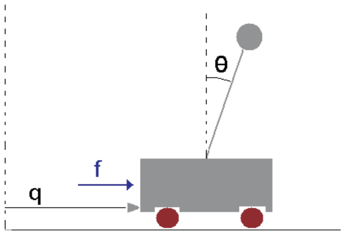

Figure 1. The inverted pendulum cart system.

applied to the cart, acting as the control input. $\beta\dot{\theta}$ is the linear damping force acting directly on the non-actuated coordinate θ[1]. δ is one structural parameter related with the mass of the cart and the pendulum, respectively [4]. After applying the following feedback

$$f = \cos\theta\sin\theta - \dot{\theta}^2\sin\theta + v(2 + \sin^2\theta + \delta)$$

into system (1), we obtain ,

$$\begin{aligned} \ddot{\theta} &= \sin\theta - \cos\theta v - \beta\dot{\theta}, \\ \ddot{q} &= v. \end{aligned} \tag{2}$$

Clearly, the above system may be expressed as:

$$\ddot{\mathbf{x}} = -F(\theta) - B\dot{\mathbf{x}} + G(\theta)u, \tag{3}$$

where

$$F(\theta) = \begin{bmatrix} -\sin\theta \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix}, \qquad G(\theta) = \begin{bmatrix} -\cos\theta \\ 1 \end{bmatrix} \tag{4}$$

and $\mathbf{x}$ stands for $\mathbf{x}^T = (\theta, q)$.

Remark 1: The latter model (3), known as a partial feedback linearization, does not retain the original mechanical structure of the **IPC,** because we canceled important nonlinearities like the Coriolis force, and is no longer a Euler-Lagrange system. This is important because it differs from the methods of Controlled Lagrangians and Controlled Hamiltonians, where the original system structure is always preserved [7]. We emphasize that canceling the nonlinearities might potentially make the matching process difficult. However, in our control strategy it is easy to accomplish the feasible energy matching conditions related to the structure of the proposed target system (closed-loop system). The target system is shaped as an asymptotic stable Euler-Lagrange system with fixed inertia matrix.

[1]As the damping force in the actuated coordinate q can be easily compensated, we do not include this term.

Comment 1: It is worth mentioning that the dissipation force can make the closed-loop system unstable [10]. In general, this force cannot be compensated by the action of a convenient control law u. However the undesirable effect of damping can be partially eliminated by using a robust stabilizing controller.

3. Control strategy

The control objective is to bring the pendulum to the up right position with the cart at the origin, assuming that the position angle of the pendulum is initialized over the horizontal plane. To this end, we propose a simple model matching for solving it. This method consists of finding one controller u that transforms the system (3) into another nonlinear system, with some desired stability properties. That is, we are looking for control law u such that the closed loop system can be written in the form

$$M_d\ddot{\mathbf{x}} = -K_d(\mathbf{x})\dot{\mathbf{x}} - \frac{\partial}{\partial x}V_d(\mathbf{x}), \tag{5}$$

where M_d and $K_d(\mathbf{x})$ are symmetric positive definite matrices. $V_d(\mathbf{x})$ is a strictly positive function[2]. Systems (3) and (5) match, for some convenient control law u, if the solution of both systems are the same. That is, ($\mathbf{x}$,u) is a solution of (3), if and only if $\mathbf{x}$ is a solution of (5). Therefore, we assure that systems (3) and (5) match, if we have

$$-F(\theta) + G(\theta)u = -M_d^{-1}K_d(\mathbf{x})\dot{\mathbf{x}} - M_d^{-1}\frac{\partial}{\partial x}V_d(\mathbf{x}). \tag{6}$$

It should be noticed that if G[3] is invertible, then we can obtain directly the desired controller u, for any given K_d and V_d. However, since G is single column then u can only act in the range space of G. This fact leads to the following constrain equation

$$0 = G^{\perp}\left(\left[M_d^{-1}\frac{\partial}{\partial x}V_d(\mathbf{x}) - F(\theta)\right] + \left[M_d^{-1}K_d(\mathbf{x})\right]\dot{\mathbf{x}}\right). \tag{7}$$

This is followed by multiplying both sides of (6) by the annihilator of G.[4] Consequently, if the unknown functions K_d and V_d are obtained for a given F, then control u can be computed directly by,

$$u = -\frac{G^T}{G^TG}\left[\left(M_d^{-1}\frac{\partial}{\partial x}V_d(\mathbf{x}) - F(\theta)\right) + M_d^{-1}K_d(\mathbf{x})\dot{\mathbf{x}}\right]. \tag{8}$$

[2]Later we discuss why (5) is asymptotically stable

[3]For simiplicity we use G to denote $G(\theta)$.

[4]Due to the fact that $G^T = [-cos\theta, 1]$, then its left annihilator is given by $G^{\perp} = \delta(x, \dot{x})[1, cos\theta]$, where δ is any strictly positive function, but for simplicity we select $\delta = 1$.

Finally, the control strategy is summarized as follows: we need to solve the matching conditions (7), which evidently can be separated in the following two matching conditions

$$0 = G^{\perp}\left[M_d^{-1}\frac{\partial}{\partial x}V_d(\mathbf{x}) - F(\theta)\right], \qquad (9)$$

and

$$0 = G^{\perp}\left[M_d^{-1}K_d(\mathbf{x})\right]\dot{\mathbf{x}}, \qquad (10)$$

Thus, control u is obtained via relation (8).

Comment 2: In this section, we do not consider the damping force effect, since it is possible to show, by using simple linear algebra that we cannot compensate this force. This means that there is no additional control variable that preserves the stability properties of the target system, and, simultaneously, assures the necessary matching condition. This is because the damping force breaks the symmetric properties of the target Lagrange or Hamiltonian System [10]. However, this effect can be partially avoided and analyzed by using simple linearization of the closed-loop system, instead of looking for a convenient Lyapunov function for the damping system.

3.1. Solving the two matching conditions:

Now our goal is to try to find the unknown matrices M_d, K_d and the unknown function V_d in order to achieve the aforementioned two matching conditions. To this end, we establish the following lemma:

Lemma 1: *If the symmetric matrices M_d^{-1} and $K_d(\theta)$ are taken as*:

$$M_d^{-1} = \begin{bmatrix} 1 & -\mu_2 \\ -\mu_2 & \mu_3 \end{bmatrix}; \qquad (11)$$
$$K_d(\theta) = \gamma M_d G(\theta) G^T(\theta) M_d,$$

where the coefficients of matrix M_d^{-1} satisfies the inequalities

$$\mu_2 > 1 \quad ; \quad \mu_{3>}\mu_2^2, \qquad (12)$$

and γ is a positive constant and function $V_d(x)$ is selected as

$$V_d(\mathbf{x}) = \frac{1}{\mu_2}\ln(-1+\mu_2) - \frac{1}{\mu_2}\ln(-1+\mu_2\cos\theta) + \frac{k_p}{2}s^2 \qquad (13)$$

where

$$s = q - \frac{\mu_3}{\mu_2}\theta + \frac{2(\mu_3-\mu_2^2)}{\mu_2\sqrt{-1+\mu_2^2}}\operatorname{arctan}h\left(\frac{1+\mu_2}{\sqrt{-1+\mu_2^2}}\tan\frac{\theta}{2}\right), \qquad (14)$$

then the two matching conditions (9) and (10) are simultaneously fulfilled, for all $\theta \in I_\mu = (-\theta_\mu, \theta_\mu)$, with

$$\theta_\mu = \cos^{-1}\left(\frac{1}{\mu_2}\right). \qquad (15)$$

Notice that, if $\theta \in I_\mu$ and $\mu_{3>}\mu_2^2$ then $K_d(\theta) > 0$ for all $\theta \in I_\mu$.

Proof: We first verify the first matching conditions related with the potential energy V_d, substituting the values of M_d^{-1} and $F(\theta)$, defined previously in the first matrix of (**??**) and the first matrix of (4) respectively, by the first matching condition (9). We have, after recalling that $G^{\perp} = (1, \cos\theta)$, the following

$$\begin{aligned} G^{\perp}\left[M_d^{-1}\tfrac{\partial}{\partial x}V_d(\mathbf{x}) - F(\theta)\right] &= \tfrac{\partial V_d}{\partial\theta}(-1+\mu_1\cos\theta) \\ &+ \tfrac{\partial V_d}{\partial q}(\mu_1 - \mu_2\cos\theta) + \sin\theta = 0 \end{aligned}. \qquad (16)$$

Easily, we can check that the following function

$$V_d(\mathbf{x}) = k_1 - \frac{1}{\mu_2}\ln(-1+\mu_2\cos\theta) + \Phi_p(s) \qquad (17)$$

is one solution of the **PDE** given in (16), where k_1 is a constant, s is an auxiliary variable given in (14), and Φ_p is any arbitrary function .[5] To guarantee that the potential energy V_d is locally positive definite, in a neighborhood of $\mathbf{x} = 0$, it is enough that,

$$V_d(0) = 0, \quad \left.\tfrac{\partial V_d}{\partial \mathbf{x}}\right|_{x=0} = 0, \quad \left.\tfrac{\partial^2 V_d}{\partial \mathbf{x}^2}\right|_{x=0} > 0. \qquad (18)$$

Applying the above conditions (18) in (17), we obtain

$$k_1 = \ln(-1+\mu_2)/\mu_2, \Phi_p^{'}(0) = 0, \Phi_p^{''}(0) > 0, \mu_2 > 1, \mu_3 > \mu_2^2. \qquad (19)$$

Thus, a convenient Φ_p is given by

$$\Phi_p(z) = \frac{k_p}{2}z^2, \qquad (20)$$

with $k_p > 0$. That is, we have validated the expression of V_d, given (13), which is strictly positive and well defined, if

$$-1 + \mu_2\cos\theta > 0. \qquad (21)$$

Evidently, the above inequality is satisfied for all $\theta \in (-\theta_\mu, \theta_\mu)$; with θ_μ defined in (15). Consequently, the proposed V_d satisfies the first matching condition, for all $\theta \in I_\mu$. That is, the proposed $K_d(\theta)$ satisfies the second matching condition.

As we can see, the aforementioned matching conditions have been easily solved. As a matter of fact, we only have solved a single **PDE** and a single algebraic equation, both related with the structure of the shaped target system. It is worth mentioning that if we employ the methodology based on the matching conditions of the controlled Lagrangians, it is necessary to solve three ordinary differential equations related to the kinetic energy shaping, and one nonlinear partial differential related with the potential energy [6], [7], [12]. A similar number of equations are needed to be solved if

[5] This **PDE** was solved by means of Mathematica^{TM} program.

we use another approach as presented in [11]. The simplicity of the two matching solutions presented here are a direct consequence of the structure of the desired closed-loop system (5). That is because the major difficulty to carry out the model matching depends largely on the ability to choose the desired closed-loop system.

Remark 2: We show that the obtained closed-loop system is also locally exponentially asymptotically stable; so, the closed-loop system is robust with respect to small unmodeled dynamics. That is, if the damping force is small enough and the system is initialized close to the origin, then, even in this case, we can expect that the system achieves the desired unstable equilibrium point. We show it with numerical simulations.

4. Closed-loop stability analysis

First of all, we note that the stability of system (3) in closed-loop with the controller u (8), is equivalent to the stability of the desired closed-loop system (5). This is a direct consequence of the definition of the matching condition. Then, the stability analysis is carried out over the desired closed-loop system (5). To this end, we propose the following Lyapunov function:

$$E(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}\dot{\mathbf{x}}^T M_d \dot{\mathbf{x}} + V_d(\mathbf{x}), \tag{22}$$

where M_d and $V_d(\mathbf{x})$ are defined in (5) and (13), respectively. Then, computing the time derivative of E, with respect to the desired closed-loop system (5), we obtain

$$\begin{aligned} \dot{E}(\mathbf{x}, \dot{\mathbf{x}}) &= \dot{\mathbf{x}}^T M_d \ddot{\mathbf{x}} + \dot{\mathbf{x}}^T \tfrac{\partial V_d(\mathbf{x})}{\partial x} \\ &= -\dot{\mathbf{x}}^T \left(K_d(\mathbf{x})\dot{\mathbf{x}} + \tfrac{\partial}{\partial x} V_d(\mathbf{x}) \right) + \dot{\mathbf{x}}^T \tfrac{\partial V_d(\mathbf{x})}{\partial x} \\ &= -\dot{\mathbf{x}}^T K_d(\theta)\dot{\mathbf{x}}. \end{aligned} \tag{23}$$

Note that the sign of E and $\dot{E}$ given in (22) and (23) are well defined, while the angle θ belongs to the set I_μ (see Lemma 1). To assure this, it is sufficient that the initial condition $(\mathbf{x}_0, \dot{\mathbf{x}}_0)$ with $\theta_0 \in I_\mu$, belonging to a neighborhood of the origin such that

$$E(\mathbf{x}_0, \dot{\mathbf{x}}_0) < V_d(\theta_\mu, 0) = C_\mu, \tag{24}$$

where θ_μ was defined previously.

Remark 3 : *The above inequality defines a stability region for the closed-loop system, that is, if the initial condition fulfills the inequality $E(x_0, \dot{\mathbf{x}}_0) < C_\mu$, with $\theta_0 \in I_\mu$. Then necessarily $\theta(t) \in I_\mu$. Given this fact, we can define a compact set Ω as:*[6]

$$\Omega = \left\{ (\mathbf{x}, \dot{\mathbf{x}}) : E(\mathbf{x}, \dot{\mathbf{x}}) < C_\mu \right\} \tag{25}$$

The set Ω has the property that all solutions of the closed-loop system (5) that start in Ω remain in Ω for ever.

Continuing with the stability analysis, we claim that the desired closed-loop system (5) is locally stable, with stability region defined by the inequality (24). Of course, the unknown M_d^{-1}, K_d and V_d have to be selected according to Lemma 1. In other words, the closed-loop solution is bounded, for any initial condition satisfying the inequality (24).

To guarantee that the closed-loop solution asymptotically converges to zero, it is necessary to use LaSalles's Theorem. To this end, we define the set

$$S = \{(\mathbf{x}, \dot{\mathbf{x}}) \in \Omega : -\dot{\mathbf{x}}^T M_d G G^T M_d \dot{\mathbf{x}} = 0\}, \tag{26}$$

and let M be the largest invariant set in S. LaSalle's theorem guarantees that every solution starting in a compact set Ω approaches M as $t \to \infty$ [13].

Let us then compute the largest invariant set M in S. From (26), it follows that

$$S = \{(\mathbf{x}, \dot{\mathbf{x}}) \in \Omega : G^T M_d \dot{\mathbf{x}} = 0\}, \tag{27}$$

which is equivalent to

$$(-\mu_3 \cos\theta + \mu_2)\,\dot{\theta} + (-\mu_2 \cos\theta + 1)\,\dot{q} = 0 \tag{28}$$

But, on set S we have that $\theta \in I_\mu$. Therefore, variables $\dot{\theta}$ and $\dot{q}$ do not change their sign, as established by the previous Lemma. Now, if variables $\dot{\theta}$ and $\dot{q}$ are different from zero and they have the same sign inside of the set S, then (θ, q) tends to go out of the invariant set Ω. This is a contradiction, because we have assumed that $(\mathbf{x}, \dot{\mathbf{x}}) \in \Omega$. Therefore, we have that $\dot{\mathbf{x}} = 0$ and $\mathbf{x}$ is a fix constant vector on set S. Now, let us define $\mathbf{x} = \overline{\mathbf{x}}$[7].Then $\overline{\mathbf{x}}$ is one of the two equilibrium points of system (3). In other words $\overline{\mathbf{x}} = (0, 0)$ or $\overline{\mathbf{x}} = (\theta = \pi, q = 0)$. However, from definitions of the invariant set Ω, given in (25), necessarily $\overline{\mathbf{x}} = 0$. Consequently, the largest invariant set $M = 0$.

Finally, we conclude that the largest invariant set M, contained in S, is constituted by the single equilibrium point $(\mathbf{x} = 0, \dot{\mathbf{x}} = 0)$. And, according to LaSalles's theorem, all the closed-loop solutions starting in Ω asymptotically converge towards the largest invariant set M, which is given by $(\mathbf{x} = 0, \dot{\mathbf{x}} = 0)$.

Summarizing the above discution, we present the main proposition of this paper:

[6] This set will be used more latter to apply LaSalle's invariance Theorem.

[7] We use the symbol $\overline{y}$ to indicate that the variable y is a constant.

Proposition 1: *Consider the non-lineal model of (3) in closed-loop with (8), where M_d,K_d and V_d are selected according to Lemma 1. Then the origin ($\mathbf{x} = \mathbf{0}, \dot{\mathbf{x}} = 0$) of desired closed loop system is locally asymptotically stable with its domain of attraction defined by the set Ω (25).*

Notice that if the constant k_p, related to the potential energy (13) is large, then domain of attraction Ω is increased. That is, we can enlarge the domain of attraction for almost all states starting above the ho

5. Simulations Results

In order to test the performance of the obtained control law we have carried out some numerical simulations using the MATLAB^{TM} system.
In the first experiment we show the influence of the control parameter k_p on the transient behavior. We test it for two different values, given by $k_p = 1$ and $k_p = 2$. We fixed the control parameters as $\mu_2 = 2.5$, $\mu_3 = 7$. The set of initial conditions were set as $\theta_0 = 1.1$[rad], $\dot{\theta}_0 = 0.1$[rad/sec], $q_0 = 0$ and $\dot{q}_0 = 0$, while the damping coefficient was fixed as $\beta = 0$. Figures 2 and 3 shows the transient behavior of the position variables and the velocity variables, respectively. As we can see, the larger the values of k_p, more oscillations are produced and all the states slowly converge to the desired equilibrium point. Intuitively, the controller injects more potential energy to the system, so that the system must dissipate all initial potential energy by means of oscillatory movements.
In the second experiment we have used the same control parameters and the same initial conditions as we did it in the previous one. To illustrate the robustness of the obtained closed-nonlinear-system, we considered a dissipative force in the unactuated direction. Figure 4 shows the closed-loop behavior of the two position variables, when the damping coefficient was fixed by $\beta = 0.1$ and $\beta = 0$. As we can see, the effect of the dissipation force tends to destabilize the closed-loop system; that is, the system converges slower to the origin. Contrarily, when the dissipation force is absent, the system converges faster to the origin.

6. Conclusions

A control strategy for the stabilization of the **IPC** around its unstable equilibrium point, assuming that the pendulum is initialized above the horizontal plane, has been presented in this paper. The control strategy is based on a partial feedback linearization of the **IPC**, followed by the application of a suitable model matching. The idea behind it is to find a feedback control law that transforms the partial feedback linearization model in a desirable target system, which has some stability properties. To carry it out, we need to solve

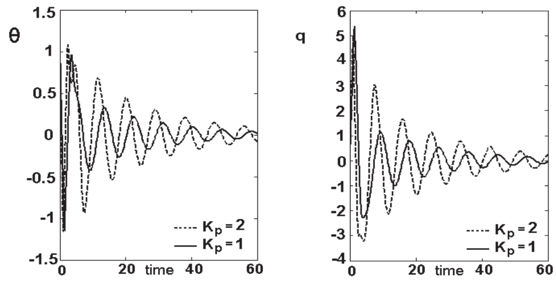

Figure 2. Closed-loop behavior of both position variables, for two different values of k_p

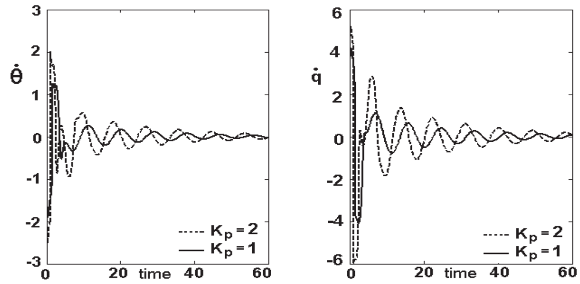

Figure 3. Closed-loop behavior of both velocities variables, for two different values of k_p

two matching conditions, both related to the structure of the desirable target system. The obtained matching conditions are very easy to solve in comparison to others matching control strategies. The resulting control strategy turns out to be locally asymptotically stable and locally exponentially stable around the origin, with a very large domain of attraction. To show that the closed-loop is locally asymptotically stable we used LaSalle's theorem and to show that it is locally exponentially stable we used a simple linearization. Consequently, the closed-loop system is robust with respect to small external forces, like the undesirable damping effect in the non-actuated coordinate.

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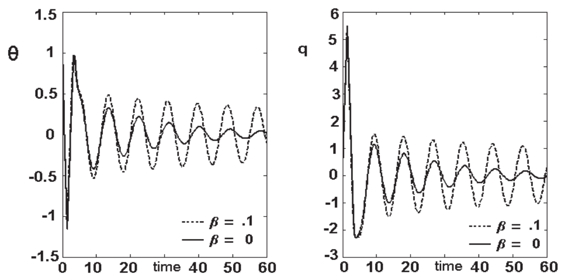

Figure 4. Robustness of the proposed controller to the damping effect.

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