Sampling Reconstruction Procedure of a Gaussian Nonstationary Process in the Presence of Jitter

V. A. Kazakov, J. A. Medina
Department of Telecommunications, ESIME-IPN, Mexico D.F., Mexico
Phone (055) 57-29-60-00 Ext. 54757 E-mail: vkazakov41@hotmail.com

Abstract

The Sampling Reconstruction Procedure (SRP) has a special interest in the communications field. There are, generally two kinds of error introduced by the sampling mechanism: errors in amplitude and errors in timing [1]. The amplitude errors could be caused by the nature of the process; in this case we will study a process which its variance depends on the time axis. The principal objective of this manuscript is to present a simple and direct method to investigate a bigger number of scenarios of the SRP for a nonstationary Markovian Gaussian process. Jitter presence is introduced to affect the sample times. This methodology is based on the conditional mathematical expectation rule. Beta function is studied as a representation of jitter distribution. The basic function is valuated at the presence of jitter. The error reconstruction function is analyzed as a comparison point between the stationary and nonstationary process, with and without jitter.

1. Introduction

The first publication devoted to the consideration of the sampling reconstruction procedure (SRP) with jitter is the Balakrishnan paper [1]. After this there were some other publications about SRP with jitter, these publications are based on different mathematical approaches. We prefer to use the statistical description with the conditional expectation rule. This rule was applied for the SRP description with jitter [2-5]. But until this time the SRP description with jitter of *nonstationary* process is not investigated.

The present paper is devoted to the investigation of SRP with jitter of the Gaussian process in the transition regime. The basic function and the error reconstruction function will be analyzed.

The model of jitter is a random variable which is represented with beta distribution [3]. The introduction of beta function allows us to consider different distributions of jitter at any sample. This characteristic works as a comparison point, and as an effective tool to analyze the effects of some variables that intervene in the process. Besides we want to study the influence of the variance changed in time and the presence of jitter for the quality of the reconstruction procedure and for the basic function.

2. General Expressions

Let us consider the general case of a Gaussian process $x(t)$ with the mathematical expectation $m(t)$, the variance $\sigma^2(t)$ and the covariance function $K(t_1, t_2)$. This is the complete information about the given process. We can write the exact expression for the multidimensional probability density function (pdf) with arbitrary m order:

$$w_m[x(t_1),...,x(t_m)] = (2\pi)^{-m/2}[\det K(t_i,t_j)]^{-1/2} \times$$
$$\times \exp\left\{-\tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[x(T_i)-m(T_i)\right]a_{ij}\left[x(T_j)-m(T_j)\right]\right\}, \quad (1)$$

where $detK(t_i, t_j)$ is the determinant of the covariance matrix

$$K(t_i,t_j) = \begin{bmatrix} K(t_1,t_1) & K(t_1,t_2) & \dots & K(t_1,t_m) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ K(t_m,t_1) & K(t_m,t_2) & \dots & K(t_m,t_m) \end{bmatrix}, \quad (2)$$

and a_{ij} are elements of the inverse covariance matrix.

When an arbitrary set of samples of Gaussian process $X,T=\{x(T_1), x(T_2), ..., x(T_N)\}$ is fixed one can find the conditional statistical characteristics of the conditional Gaussian process: the conditional mathematical expectation $\tilde{m}(t)$ and the conditional variance $\tilde{\sigma}^2(t)$. They are determined by the following known formulas [6]:

$$\tilde{m}(t) = m(t) + \sum_{i=1}^{N}\sum_{j=1}^{N} K(t,T_i)a_{ij}\left[x(T_j)-m(T_j)\right], \quad (3)$$

$$\tilde{\sigma}^2(t) = \sigma^2(t) - \sum_{i=1}^{N}\sum_{j=1}^{N} K(t,T_i)a_{ij}K(T_j,t). \quad (4)$$

As we can see, the error reconstruction function does not depend on the set of samples *X*, but depends on the time location *T*.

If we consider a process with $m(t)=0$, from (3), we can get the expression of the reconstruction process that just depends of the summation of the product of each sample $x(T_j)$ by the function named *basic function* $B_j(t)$:

$$\hat{x}(t)=\sum_{i=1}^{N}\sum_{j=1}^{N}K(t,T_i)a_{ij}\left[x(T_j)\right]=\sum_{j=1}^{N}x(T_j)B_j(t), \quad (5)$$

$$B_j(t)=\sum_{i=1}^{N}K(t-T_i)a_{ij}. \quad (6)$$

The basic function depends principally on the behavior of the covariance function $K(t_1,t_2)$, and exist one basic function for each sample.

3. Model of Jitter

We have analyzed in the previews paragraphs *the conditional mathematical expectation rule* to describe the *SRP* of the stochastic processes without jitter. This rule is enough when we know exactly the sample times $T=\{T_1,T_2 \ldots,T_N\}$. But, when the sample times are random, $\breve{T}_i=\{\breve{T}_1,\breve{T}_2,\ldots,\breve{T}_N\}$ is necessary to apply the statistical average to this rule with respect to the jitter pdf of each sample $w(\breve{T}_i)$ in order to get statistical characteristics of SRP.

In (3) and (4) we can see that there are three time arguments: (t,T_i), (T_i,T_j) and (T_j,t). In the presence of jitter these arguments are the sample instant turned into random variables: $(t,\breve{T}_i)$, $(\breve{T}_i,\breve{T}_j)$ y $(\breve{T}_j,t)$. Notice that the first and third lapse depend only on one random sampling instant $\breve{T}_k=\{T_k+\varepsilon_k\}$, where ε_k represents an arbitrary variable $(k = i, j)$ and T_k is a constant. So, then the pdf of each random sampling instant depends on the jitter pdf of each sample $w(\varepsilon_k)$:

$$w(\breve{T}_k)=w(T_k+\varepsilon_k). \quad (7)$$

In order to introduce different types of jitter pdf in an easy way, we will use beta distribution [3]. The principal advantage of this methodology is that we just need to change two parameters to get a great diversity of distributions that have all the characteristics of a statistical distribution.

However, the classic representation of beta distribution has a problem, that is because it is just defined by the limits $0<x<1$, furthermore, the jitter exist normally in both sides of the sample time. For these reasons is necessary to rewrite it for whichever time interval:

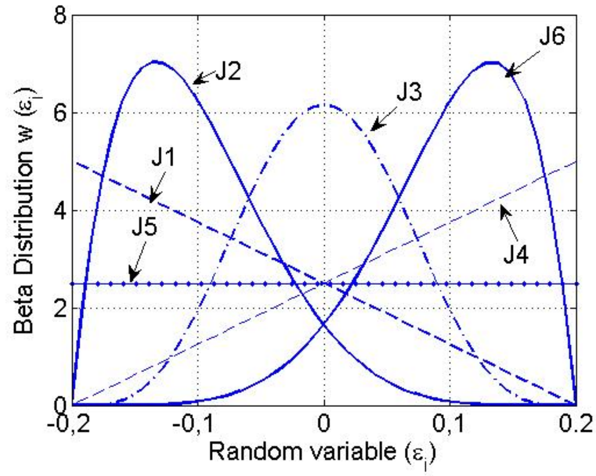

Fig.1. Different jitter pdf. In this case, the distributions are within the symmetric interval [-0.2; 0.2].

$$w(\varepsilon_i)=\frac{(\varepsilon_i-a_i)^{\beta-1}(b_i-\varepsilon_i)^{\beta-1}}{B(\beta,\gamma)(b_i-a_i)^{\beta+\gamma-1}}, \quad a_i\le\varepsilon_i\le b_i. \quad (8)$$

With the last expression, the pdf of jitter can be represented by different forms, depending on the value of β and γ. The different shapes of the beta distribution are presented in Fig. 1.

It is important to say that the Gaussian and Rayleigh distributions are truncated, that is because they are limited within the restricted interval *[a,b]*

Beta distribution is characterized by the mathematical expectation $\langle\varepsilon_i\rangle$ and variance $\langle\varepsilon_i^{2}\rangle$:

$$\langle\varepsilon_i\rangle=a_i+\frac{\beta}{\beta+\gamma}(b_i-a_i), \quad (9)$$

$$\left\langle\dot{\varepsilon}_i^{\,2}\right\rangle=\frac{\beta\gamma\left(b_i-a_i\right)^2}{\left(\beta+\gamma\right)^2\left(1+\beta+\gamma\right)}. \quad (10)$$

4. Model of a nonstationary process

Let us consider a process *x(t)* on the output of an integrated *RC* circuit (with the parameter $\alpha=1/RC$) driven by white noise *n(t)* with the constant spectral density $N_0/2$. Such an output process is described by the stochastic differential equation of the first order with the constant coefficients:

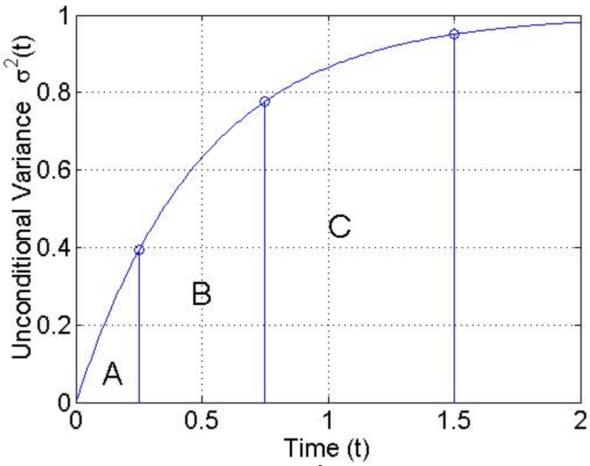

Fig. 2. Unconditional variance σ^2 *(t)* delimited in three areas of interest.

$$\dot{x}(t) = -\alpha x(t) + \alpha n(t). \tag{11}$$

The required covariance function in the transition regime is determined by the following formulas:

$$K(t, t+\tau) = \sigma^2(t)\exp(-\alpha|\tau|), \tag{12}$$

$$\sigma^2(t) = 1 - \exp(-2\alpha t). \tag{13}$$

Now, with this method, it is easy to concretize the expressions for the reconstruction function (3) and for the error reconstruction function (4) by introducing (12) and (13) and defining the variance interval in the process. Fig. 2 that shows the graphic of (13) divided at different intervals to be valuated in (4) and (6). The results of this operation for different intervals (A, B y C) are presented in Fig. 3 and Fig. 4.

Because the process *x(t)* is Markovian, then the interpolation procedure depends only on the two samples $x(T_i)$ and $x(T_{i+1})$; the extrapolation depends only on the unit sample $x(T_n)$ [4].

Fig. 3 shows the error basic function for some different intervals of variance without the presence of jitter. To make this concept easier to understand, one can imagine that in the area A of Fig. 2 that goes from t=0 to 0.25 we introduce the five samples of the Fig. 4 without jitter, therefore, the variance for each sample depends on its location in the time axis.

The sampling interval is equal for all the cases in Fig. 3 and Fig. 4. We can note that the error is zero at the points of the samples. Is well known that the interpolation error function of a stationary process does not depend on the location of the discretization interval in the time axis,

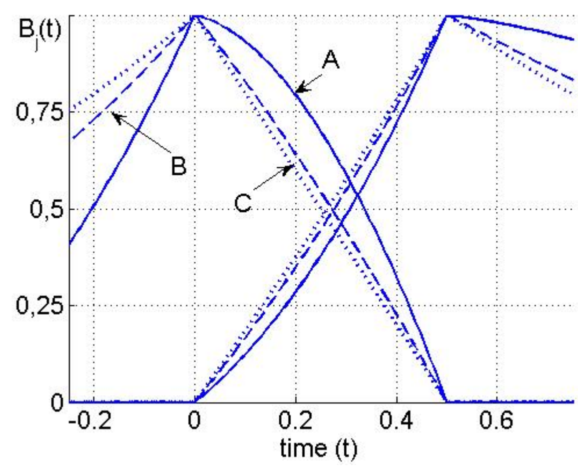

Fig. 3. Basic function *Bj(t)* valuated for some different intervals of variance.

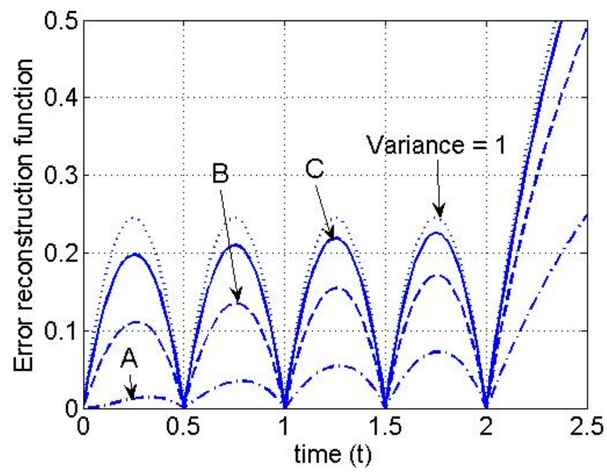

Fig. 4. Error reconstruction functions of a first order filter for some different intervals of variance.

but depends only on its duration. As we can see, in this case, the error reconstruction function and the basic function depend on the variance, and on the variance speed, for example in A, variance has an increment of 0.4 in just 0.25 seconds (see Fig. 2), while C has an increment of 0.17 in 0.75 seconds. It is possible to make a comparison with the stationary regime and with the case of the constant variance. Furthermore it is easier to observe in this case that the peaks of error have not the same values unless this is a Markovian process. This effect is due to the change of the variance with respect to the time. Therefore, it is possible to say that the error is proportional to the variance.

5. The SRP in the presence of jitter at the nonstationary process.

When we know the pdf of each random sampling instant $w(\check{T}_i)$ $(i=1,\ldots, N)$, is possible to define the general expressions of the average of the error reconstruction function $\left\langle \tilde{\sigma}^2(t) \right\rangle$, and the basic function $<B_j(t)>$:

$$\left\langle B_j(t) \right\rangle = \left\langle \sum_{i=1}^{N} K(t-\breve{T}_i) a_{ij} \right\rangle_{\breve{T}_i} . \qquad (14)$$

$$\left\langle \tilde{\sigma}^2(t) \right\rangle = \sigma^2(t) - \left\langle \sum_i \sum_j K\left(t, \breve{T}_i\right) a_{ij} K\left(\breve{T}_j, t\right) \right\rangle_{\breve{T}_i, \breve{T}_j} , \quad (15)$$

the angular parenthesis means the statistic average operation with respect to the random variables below the parenthesis. The covariance matrix now is:

$$\left\langle K(\breve{T}_{i,} \breve{T}_j) \right\rangle = \begin{bmatrix} \left\langle K(T_{1,} T_1) \right\rangle & \cdots & \left\langle K(T_{1,} T_n) \right\rangle \\ \vdots & \ddots & \vdots \\ \left\langle K(T_{n,} T_1) \right\rangle & \cdots & \left\langle K(T_{n,} T_n) \right\rangle \end{bmatrix}, \quad (16)$$

We can see from (16) that the statistic average is applied to every function due to the random variables which describe the random instants of sampling.

Fig. 5 shows the basic function affected by jitter. At the first sample the jitter pdf is symmetric with respect to de sample moment as we can see from Fig. 1 (see distribution J5). With symmetrical jitter, every basic function has a maximum located at the original position of the sample. It is possible to note that exists an asymmetry in the line pointed by the arrow AJ5, this is due to the fast change of variance (see area A in Fig. 2). The second sample is affected by asymmetric jitter to the right. That is, it is possible to observe that the pick of the basic function has a displacement to the right. This effect depends on the distribution of jitter, for example, J2 from Fig. 2 has a bigger mathematical expectation to the left than J1.

Fig. 6 shows the error reconstruction function average of the four cases where the covariance interval are: A)[0;0.25]; B)[0.25; 0.75]; C) [0.75; 1.5]. The samples affected by jitter are the first sample at $t=0$, the third sample at $t=1$, and the fifth sample at $t=2$, the jitter pdf respectively introduced is J3, J4 and J2 that is shown in Fig. 2. The width of every jitter interval is 0.2 and is symmetric with respect of the sample time.

We comment the shapes of the curves on Fig. 6:

The first is that the influence of the jitter is really evident just comparing Fig. 4 and Fig. 6, because the action of the variance is increased.

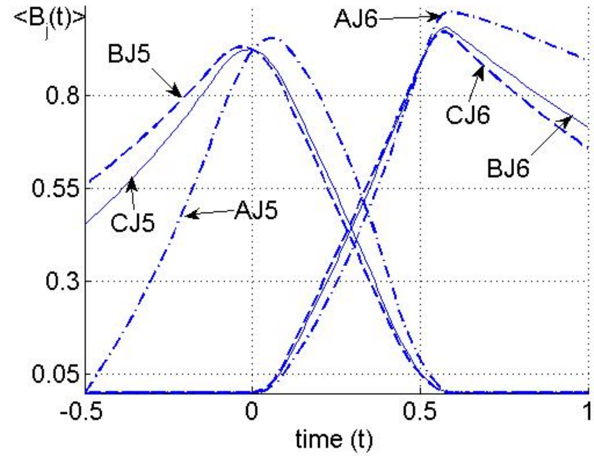

Fig.5. Average of the basic functions when two samples with jitter take part in the Markovian process.

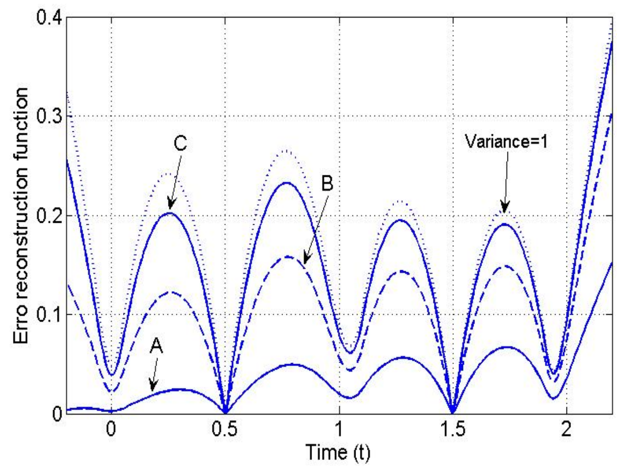

Fig. 6. Error reconstruction functions of a first order RC filter, with nonstationary variance, and the presence of jitter.

Second, the error at the sample time is not equal to zero, but depends on the width of the jitter interval.

The third effect is that, when we apply asymmetric jitter pdf at any sample, the minimum of the error is no centered at the sample time, if not at the right or left of the sample depending of the kind of the introduced asymmetry.

It is possible to determine the point of the minimum error with the mathematical expectation of the jitter pdf $<\varepsilon_i>$ by (9).

In Fig. 6 a no common asymmetry is observed in the curve marked with the arrow A that is caused by the fast increment of the variance (see Fig. 2).

Finally, the displacement caused by the jitter causes an increase or decrease of the maximum error because the sample comes closer or it takes away from the near samples. For example the mathematical expectation of the sample 3 located at $t=1$ is displaced to the right, this causes that the interval between sample 2 and sample 3 decreases and the error decreases too. In the other hand, this displacement causes that the distance between the sample 3 and sample 4 increases, and therefore, the error between the interval gets bigger.

6. Conclusions

The proposed methodology describes us in a simple and complete way the effects of the jitter in the whole domain of the time. For the same thing, this methodology is advisable since it has allowed us to consider different scenarios with jitter and with the nonstationary variance.

The reconstruction error is not zero in the presence of jitter. The minimum of the error depends of the mathematical expectation of the jitter.

Introduction of jitter makes the uncertainty to grow up, so then the error has an increment or a displacement that depend on the distribution of the jitter.

The increment of variance means an increment of error reconstruction due to the process becomes more chaotic and the uncertainty is bigger.

7. References

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